Field Profile in a Single-Mode Curved Dielectric Waveguide of Rectangular Cross Section

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Abstract—**An approximate and simple method for predicting the field profile in a curved dielectric waveguide of rectangular cross section is described. For a single-mode propagation, it is shown that the transverse field can be approximated inside the dielectric guide by the Airy function of the first kind and that the radial attenuation constant is a function of the bending radius outside the guide. Experimental verification of the theoretical results is included.**

I. Introduction

Recently, dielectric waveguides have been used as transmission lines for millimeter-wave integrated circuits. These lines are often curved to satisfy the small-size requirements of the system packaging. Also, many of the passive dielectric components, such as ring resonators and couplers, are comprised of curved waveguide sections. Therefore, it is necessary to have a good understanding of the behavior of the field in the neighborhood of the bend in order to accurately predict the coupling and radiation characteristics.

It is well known that the transverse field profile in a curved dielectric guide is not symmetric about the center of the guide. The field maximum is shifted outwardly, and there exists a significant portion of energy on the outer side of the curved guide. On the inner side, the field radially decays much faster than that of a straight guide. Therefore, unless the radius of curvature is very large, the usual assumption that the field in the curved guide can be approximated by that of a straight guide may not be valid.

The field deformation of a step-index curved optical fiber of circular cross section has been investigated by Marcuse and others [1]–[3]. For a dielectric waveguide of rectangular cross section, Marcatili's approach did include the outward shift of the field [4]. However, his analysis is rather involved and it is often useful to seek simple solutions which represent the field behavior in a curved dielectric waveguide in an approximate but accurate manner.

In this approximate solution, the field in the dielectric core in the curved waveguide is represented in terms of the solutions of the wave equation in an inhomogeneous linear medium via a conformal mapping [5], [6]. In the air medium outside of the dielectric waveguide, the radial attenuation constant is modified from that of the straight guide because of the curvature. The validity of these approximate expressions is verified by experiments carried out at X-band. The junction discontinuity and the discrete-beam radiation at the transitions between the curved and the straight guides are ignored in this analysis.

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II. Analysis of the Transverse Field Variation

A. Fields in the Dielectric Medium

Consider a curved dielectric waveguide with a homogeneous refractive index n_1 surrounded by air, as shown in Fig. 1. The analysis is based on a conformal mapping which will be used to transform a homogeneous step-index curved waveguide into a straight guide with an inhomogeneous refractive index. The cylindrical plane (r,θ) is mapped into the linear (w,z) plane by the transformation

$$\begin{cases} w = R\ln\left(\frac{r}{R}\right) \\ z = R\theta \end{cases} \tag{1}$$

where

$$r = R + x \tag{2}$$

and w and z are transverse and longitudinal coordinates of the linear plane, respectively (see Fig. 2). According to these figures, the longitudinal propagation constant in the transformed straight guide is constant throughout the cross section of the guide as if obtained from a straight guide itself [7], whereas in the original curved structure, the wavelength varies as a function of r.

For a moderate-to-large curvature, the transverse refractive index in the (w,z) plane can be expressed as [3]

$$n(w) = n_1\exp(w/R) \simeq n_1\left(1+\frac{w}{R}\right). \tag{3}$$

Obviously, with this refractive index profile (Fig. 3), the maximum in the energy distribution will shift toward the outer edge of the curved guide, and more energy is concentrated near the outer dielectric–air interface than that in a straight guide. The degree of shift depends on the bending radius R.

The wave equation for the electric field E in the dielectric waveguide in the (w,z) plane becomes

$$\nabla^2E(w,y,z,) + n^2(w)k_0^2E(w,y,z) = 0 \tag{4}$$

where k_0 is the free-space wavenumber, and w is the

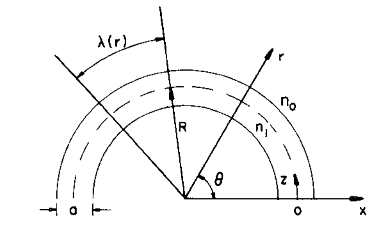

Fig. 1. Curved dielectric waveguide.

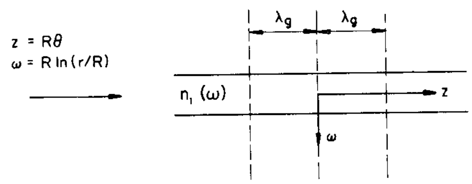

Fig. 2. Transformed straight guide in the (w,z) plane.

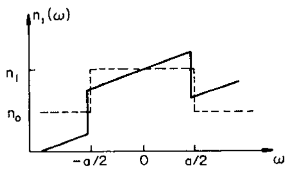

Fig. 3. Transverse index profile in the (w,z) plane.

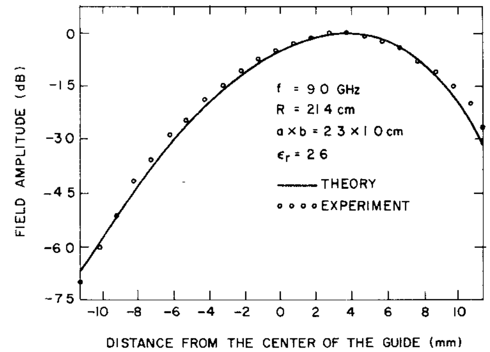

Fig. 4. Field amplitude as a function of the transverse position in the guide.

transverse coordinate in the (w,z) plane.

Since the y-variation in a dielectric guide of rectangular cross section is invariant under the transformation, the transverse propagation constant k_y is the same as that for a straight guide. Also, from the assumption that the longitudinal propagation constant is uniform throughout the cross section of the transformed guide, the wave equation (4) reduces to a scalar problem involving only the w-variation component of the field

$$\frac{d^2}{dw^2}\psi(w)+\left(\epsilon_1 k_0^2-k_y^2-k_g^2+\frac{2\epsilon_1 k_0^2}{R}w\right)\psi(w)=0 \quad (5)$$

where k_y and k_g are the transverse and axial propagation constants of the straight guide and $\epsilon_1=n_1^2$.

The Wentzel–Kramer–Brillouir (WKB) approximation can be used to solve (5). However, by introducing a new variable given by

$$\zeta=\left(k_g^2+k_y^2-\epsilon_1 k_0^2\right)\left(\frac{2\epsilon_1 k_0^2}{R}\right)^{-2/3}-\left(\frac{2\epsilon_1 k_0^2}{R}\right)^{1/3}w \quad (6)$$

the nonlinear differential equation for $\psi(w)$ may be written as

$$\psi''(\zeta)-\zeta\psi(\zeta)=0. \quad (7)$$

The linear independent solutions for this equation are given by

$$\mathrm{Ai}(\zeta) \qquad \mathrm{Bi}(\zeta)$$

and

$$\mathrm{Ai}(\zeta) \qquad \mathrm{Ai}\left(\zeta e^{\pm 2\pi j/3}\right). \quad (8)$$

The choice for the combination of these solutions for representing the field in the curved dielectric waveguide is not obvious. Given that the field maximum in a curved dielectric guide always shifts toward the outer edge and knowing the behavior of the Airy functions [8], the solution ψ can be approximately represented as

$$\psi(\zeta)\doteq C\mathrm{Ai}(\zeta) \quad (9)$$

where C is an arbitrary normalizing constant, and ζ is related to the original transverse parameter x by (1), (2), and (6).

At the transition region at the beginning of the bend, some radiation is emitted from the guide in the form of discrete beams. Also, the field distribution travels in a zigzag down the guide [9]. However, an equilibrium is soon reached with the steady-state field distribution being offset at a fixed distance from the center of the guide. Since the transition between the curved and the straight guides is very gradual, this junction discontinuity should be insignificant and is not analyzed here.

The fundamental mode in the straight dielectric guide section is the E_{11}^y mode and is launched by a conventional metal waveguide that supports a TE_{10} mode. The field amplitude in the curved region was plotted as a function of the traverse position x in the guide according to (9) and compared with the experimental data, as shown in Fig. 4. The experimental field amplitude was recorded by an electric field probe with the electric field polarized normal to the plane of the curvature.

In many applications, the knowledge of the position of the field maximum is necessary. For instance, in an E-plane coupled structure that consists of a straight and a curved dielectric guide of rectangular cross section with the electric field vertically polarized [10], the strongest coupling coefficient is realized when the field maxima in both guides almost coincide. To obtain the offset distance from the center of the guide of the field maximum in a single-mode curved dielectric waveguide, the argument of the field component in (9), ζ, is set equal to -1.0025. At this value, the Airy function of the first kind assumes a maximum. The offset distance of the field maximum from the center of the original curved guide is then obtained through w using (1), (2), and (6). Figs. 5 and 6 are the plots of the shift of the field maximum from the center of a curved dielectric waveguide as a function of the mean radius and the frequency, respectively. The vertical bars are the experimental results since it is very difficult to record the exact

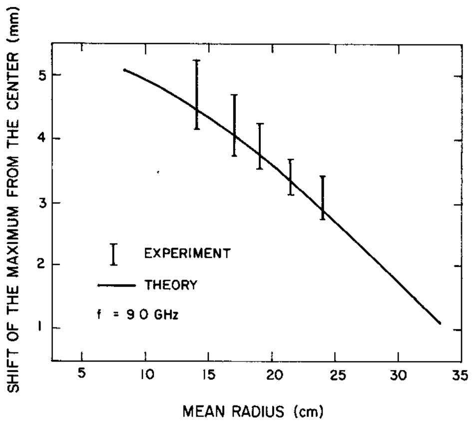

Fig. 5. Shift of the field maximum from the center of the guide versus mean radius.

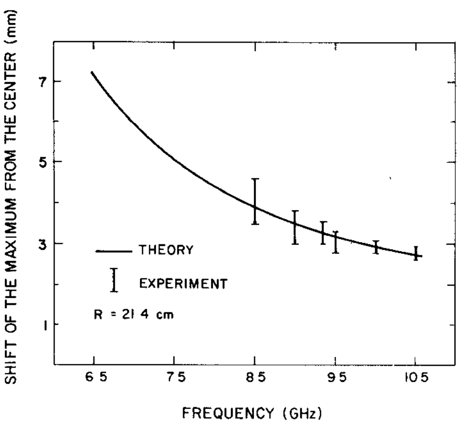

Fig. 6. Shift of the field maximum from the center of the guide versus frequency.

position of the field maximum. The vertical bar would represent a neighborhood within which the maximum would take place.

It is important to point out that if the bending radius is decreased significantly to a point where $n_0(1+a/2R)$ equals or exceeds the effective refractive index of the guide, and when a caustic is formed within the dielectric guide, this analysis will fail. Here, the effective refractive index is defined as the ratio of the longitudinal propagation constant of a straight guide to that of free-space wavenumber. In the first case, all guidance ceases and the radiation loss rises to infinity. For the latter, the analysis changes to a single air-dielectric boundary problem. These cases are discussed in [11]. When the curved radii become very large, i.e., $R \to \infty$, the refractive index approaches a constant value n_1 throughout the guide as indicated by (3). The analysis then reduces to that for a straight guide.

B. Fields in the Air Medium

In the air medium outside of the dielectric core, it is possible to obtain an approximate transverse field profile by using the same analysis as described in the previous section. On the inner side of the curved guide, experimental results have shown that the x-variation component of the field can assume the Airy function of the first kind (Ai). On the outer side of the curved guide, however, the Ai function alone will not satisfy the characteristics of the field in this region. A satisfactory combination of all the Airy functions to represent the field in this region has not yet been found. A different approximate approach may be necessary.

In a curved dielectric waveguide, the bending of the guide is likely to lead to a distortion of the phase fronts of the wave being guided along the curved guide as can be seen in Fig. 1. Since the phase velocity, which is parallel to the curved surface, is higher in the direction of increasing r, the radial dependence of the guided propagation constant in the region close to the guide can be approximated by [12]

$$k_z(r)=k_g\left(\frac{R}{r}\right). \tag{10}$$

The expression for the radial attenuation constant outside the curved dielectric waveguide is then approximated by the generalized effective dielectric constant technique [13]

$$\xi^2(r)=k_z^2(r)-k_0^2=k_g^2\left(\frac{R}{r}\right)^2-k_0^2. \tag{11}$$

This equation suggests that the outward attenuation constant in the air medium is no longer a constant but varies as a function of the distance away from the guide. On the outer side of the bend, ξ is real and positive in the region close to the dielectric guide, which implies that the field decays exponentially away from the guide but is still guided. In this region, according to the new attenuation constant, the radial variation of the field can be expressed as

$$\psi_0(x)\sim\exp\left\{-\int_{a/2}^{x}\left(k_g^2\left(\frac{R}{R+\tau}\right)^2-k_0^2\right)^{1/2}d\tau\right\}. \tag{12}$$

However, at a critical distance x_{cr} away from the guide, the phase velocity of the guided wave equals that of free space. The field disassociates itself from the guided mode and radiates into space. This phenomenon is sometimes described as the electromagnetic tunnelling [3], [11]. When the field becomes a radial traveling wave, the concept of an effective dielectric constant becomes meaningless. Therefore, far from the guide, the field takes the form

$$\psi_{0_+}(x)=\psi_0(x_{cr})\,H_v^{(2)}(K_0(R+x)) \tag{13}$$

where

$$v=k_gR. \tag{14}$$

The plus sign on ψ_{0_+} indicates that the field is only valid in the region where $x \geqslant X_{cr}$. The choice of the Hankel function of the second kind is dictated by the requirement that the field must become an outward traveling wave of $x >> x_{cr}$[12]. Since the exact position of x_{cr} is impossible to predict, it will be arbitrarily defined as the distance x where

$$\psi_0'(x)=\psi_{0_+}'(x) \tag{15}$$

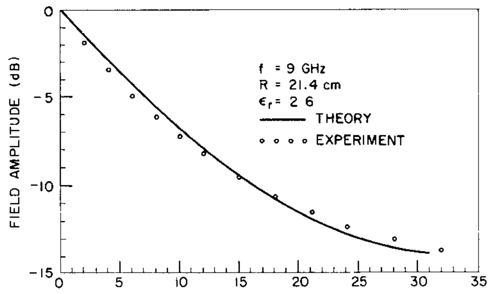

Fig. 7. Field amplitude versus distance from the guide on the outer side of a curved dielectric waveguide.

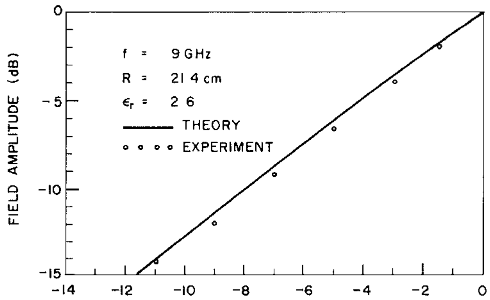

Fig. 8. Field amplitude versus distance from the guide on the inner side of a curved dielectric waveguide.

where the prime indicates differentiation with respect to x. For sufficiently large R

$$x_{\mathrm{cr}} \doteq R\left(\frac{k_g}{k_0}-1\right). \qquad (16)$$

The field amplitudes derived from (12), (13) were plotted together and compared with experimental results, as shown in Fig. 7. The discrete-beam radiation effect reported by others [5] was not apparent in this experiment.

On the inner side of the bend, the radial attenuation constant ξ is always real which indicates that the field must decay in the direction of decreasing r. With ξ given in (11), the field profile in the inner side of the bend becomes

$$\psi_I(x) \sim \exp\left\{\int_{-a/2}^{x}\left(k_g^2\left(\frac{R}{R+\tau}\right)^2-k_0^2\right)^{1/2} d\tau\right\} \qquad (17)$$

where x is negative in the region $r<R$. The experimental and theoretical results for the field amplitude in the inner side of the curved dielectric waveguide are plotted in Fig. 8.

III. Conclusion

In this paper, we have presented an approximate method for predicting the field profile in a curved dielectric waveguide of rectangular cross section. For simplicity, the dielectric and free-space regions were treated separately. Inside the dielectric waveguide, the transverse field can be expressed as an Airy function via a conformal transformation. On the outer side of the guide, the radial attenuation constant is not a constant and has real value near the guide. Far away from the guide, the field is an outward traveling wave and is expressed in terms of the Hankel function of the second kind. On the inner side of the bend, the field decays strictly exponentially. Experimental verification of the field behavior predicted by the approximate formulas is included. This analysis can be applied to other guiding structures of rectangular cross section where the transverse propagation constants can be obtained independently.

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